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THE GENERALIZED FORMULA FOR AGGREGATIVE PRICE INDICES

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ABSTRACT

In the paper we propose a generalized formula for aggregative price indexes which satisfies most of the postulates coming from axiomatic price index theory. It is shown, that the ideal Fisher index and Lexis index are special cases of the proposed formula. Moreover, using the generalized formula we can easily define new indexes, which also satisfy given postulates.

Key words: aggregative price indexes, Laspeyres index, Paasche index, Fisher index, Lexis index.

1. Introduction

The contemporary economics makes use many statistical indexes in order to calculate the dynamics of growth of prices or quantities. The indexes make it possible to compare two periods of observations of the given economic processes. For example, the production in next periods depends on the former growth of prices, quantities and the profitability. Hence, it is very important to use the proper and well constructed indexes (see Fisher (1972)). The history of aggregative indexes is quite long - Laspeyres and Paasche indexes have been known since 19-th century (see Diewert (1978, Shell (1998)). Depending on the type of an economic problem we may also use one of the following indexes: Fisher ideal index (see Fisher (1922)), Törnqvist index (Törnqvist (1936)), Lexis index and other indexes (see Zając (1994), von der Lippe (2007)). The statistical indexes are very useful in economy (see Moutlon (1999), Seskin (1998)). Balk (1995) wrote about axiomatic price index theory, Diewert (1978) showed that the Törnqvist index and Fisher ideal index approximate each other. But it is really hard to indicate the best one of the statistical indexes (Dumagan (2002)). The choice of index depends on the information we want to get. If we are interested in dynamics of money in time we should use Fisher or Lexis indexes (see Zając (1994)).

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From the theoretical point of view, a good index should satisfy a group of postulates (tests), coming from the axiomatic index theory (see Luszniewicz (1984), Balk (1995)). In this place we only mention about the following postulates: *linear homogeneity, proportionality*, i*dentity, time reversibility, circularity, commensurability* and *determinantness test* (see Luszniewicz (1984), Białek (2005), von der Lippe (2007)). Not every statistical index satisfies the mentioned tests. A system of minimum requirements of the index comes from Marco Martini (1992). According to the mentioned system, the price index should at least satisfy the following three conditions: *identity*, *commensurability* and *linear homogeneity.*

Let us consider a group of N components observed at times s , t and let us denote:

$P^s = [p_1^s, p_2^s, ..., p_N^s]$ - the vector of components prices at time s ;

$P^t = [p_1^t, p_2^t, ..., p_N^t]$ - the vector of components prices at time t;

$Q^s = [q_1^s, q_2^s, ..., q_N^s]$ - the vector of components quantities at time s ;

$Q^t = [q_1^t, q_2^t, ..., q_N^t]$ - the vector of components quantities at time t.

We consider only price indexes but all the discussion can be generalized to the case of quantity indexes. One of the most popular and often used price indexes are Paasche and Laspeyres indexes. Using the above notations the Paasche price index can be defined as follows:

$$I_{Pa}^p(Q^t, P^s, P^t) = \frac{Q^t \circ P^t}{Q^t \circ P^s}, \tag{1}$$

and Laspeyres price index:

$$I_L^p(Q^s, P^s, P^t) = \frac{Q^s \circ P^t}{Q^s \circ P^s}, \tag{2}$$

where "$\circ$" denotes a scalar product of two vectors.

As it is known, the indexes (1) and (2) do not satisfy some of the postulates for price indexes.

For example, the *time reversibility* for the price index I^p, described by formula (3)

$$I^P(Q^s, Q^t, P^s, P^t) = \frac{1}{I^P(Q^t, Q^s, P^t, P^s)}, \tag{3}$$

is not satisfied. Fisher proposed another definition based on Paasche and Laspeyres formulas (see Fisher (1922)). His formula I_F^p is a geometric mean of Paasche and Laspeyres indexes, it means:

$$I_F^p(Q^s,Q^t,P^s,P^t)=\sqrt{I_L^p(Q^s,P^s,P^t)I_{Pa}^p(Q^t,P^s,P^t)}\,. \tag{4}$$

The Fisher definition is called an "ideal formula", because it satisfies the tests mentioned above including *time reversibility*. Now we present a general class of price indexes.

2. The generalized formula for aggregative price indexes

Let $f_j : R_+^N \times R_+^N \to R_+^N$, for $j=1,2,...,m$ with some established $m\in N$, be such functions that for any $X,Y,Z\in R_+^N$ it holds

$$\sum_{j=1}^{m}\ln\frac{f_j(X,Y)\circ Z}{f_j(Y,X)\circ Z}=0, \tag{5}$$

or equivalently

$$\prod_{j=1}^{m}\frac{f_j(X,Y)\circ Z}{f_j(Y,X)\circ Z}=1. \tag{6}$$

Certainly the set of functions satisfying (6) is not empty – for example we could assume $f_j(X,Y)=c_j(X+Y)$ for $j=1,2,...,m$ and some $c_j\in R_+$.

Let us also notice that the condition (6) means equivalently that for any $Q^s,Q^t,P^s,P^t\in R_+^N$ it holds

$$\prod_{j=1}^{m}\frac{f_j(Q^s,Q^t)\circ P^t}{f_j(Q^t,Q^s)\circ P^t}=1,\ \prod_{j=1}^{m}\frac{f_j(Q^s,Q^t)\circ P^s}{f_j(Q^t,Q^s)\circ P^s}=1. \tag{7}$$

We propose the following, generalized formula for price indexes:

$$I^p(Q^s,Q^t,P^s,P^t)=[\prod_{j=1}^{m}\frac{f_j(Q^s,Q^t)\circ P^t}{f_j(Q^s,Q^t)\circ P^s}]^{\frac{1}{m}}. \tag{8}$$

Let us signify by $\Lambda(m)$ the class of price indexes defined by (6) and (8) for some established $m\in N$. It can be easily proved, that for any $k\in N$ we have (compare (28)-(31)):

$$I^P(Q^s,Q^t,P^s,P^t)\in\Lambda(m)\Rightarrow I^P(Q^s,Q^t,P^s,P^t)\in\Lambda(km). \tag{9}$$

Moreover, each index from the class $\Lambda(m)$ satisfies most of the mentioned postulates. In particular, the following theorem is true:

Theorem 1

Let I^P be the aggregative price index with the structure described by (8) with additional condition (6). Then, the I^P index satisfies: *linear homogeneity*, *identity, proportionality, time reversibility* and *determinantness test*. If we assume additionally that for each f_j it holds $f_j(\lambda X, \lambda Y) = \lambda f_j(X, Y)$, then we also have *weak commensurability* [1] satisfied.

The proof is quite easy. For example let us notice, that for some $\lambda > 0$ we have:

$$I^p(Q^s, Q^t, P^s, \lambda P^t) = [\prod_{j=1}^{m} \frac{f_j(Q^s, Q^t) \circ (\lambda P^t)}{f_j(Q^s, Q^t) \circ P^s}]^{\frac{1}{m}} =$$

$$= [\lambda^m \prod_{j=1}^{m} \frac{f_j(Q^s, Q^t) \circ P^t}{f_j(Q^s, Q^t) \circ P^s}]^{\frac{1}{m}} = \lambda I^p(Q^s, Q^t, P^s, P^t). \tag{10}$$

Thus, the *linear homogeneity* is satisfied. The *determinantness test* is also satisfied because we always have $I^p > 0$. Moreover, let us notice that using (7) and (8) we get

$$I^p(Q^t, Q^s, P^t, P^s) \cdot I^P(Q^s, Q^t, P^s, P^t) =$$

$$= [\prod_{j=1}^{m} \frac{f_j(Q^t, Q^s) \circ (P^s)}{f_j(Q^t, Q^s) \circ P^t}]^{\frac{1}{m}} [\prod_{j=1}^{m} \frac{f_j(Q^s, Q^t) \circ (P^t)}{f_j(Q^s, Q^t) \circ P^s}]^{\frac{1}{m}} =$$

$$= [\prod_{j=1}^{m} \frac{f_j(Q^t, Q^s) \circ P^s}{f_j(Q^s, Q^t) \circ P^s} \cdot \frac{f_j(Q^s, Q^t) \circ P^t}{f_j(Q^t, Q^s) \circ P^t}]^{\frac{1}{m}} =$$

$$= [\prod_{j=1}^{m} \frac{f_j(Q^t, Q^s) \circ P^s}{f_j(Q^s, Q^t) \circ P^s}]1^{\frac{1}{m}} [\prod_{j=1}^{m} \frac{f_j(Q^s, Q^t) \circ P^t}{f_j(Q^t, Q^s) \circ P^t}]1^{\frac{1}{m}} = 1^{\frac{1}{m}} = 1 \tag{11}$$

Thus, *the time reversibility* (3) is satisfied.

Certainly, if $P^s = P^t$ then $I^p(Q^s, Q^t, P^s, P^t) = 1$ - thus the *identity* is satisfied (by implication also the *proportionality*). The proof of *weak commensurability* is omitted.

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[1] It means that $I^p(\lambda^{-1}Q^s, \lambda^{-1}Q^t, \lambda P^s, \lambda P^t) = I^p(Q^s, Q^t, P^s, P^t)$.

3. Special cases of the generalized formula

First of all let us notice, that in the special case of $m = 2$ and for

$$f_1(X,Y) = X\,, \tag{12}$$

$$f_2(X,Y) = Y\,, \tag{13}$$

where $X = (x_1, x_2, .., x_N)$, $Y = (y_1, y_2, .., y_N)$, we have for any $Z \in R_+^N$

$$\prod_{j=1}^{2} \frac{f_j(X,Y)\circ Z}{f_j(Y,X)\circ Z} = \frac{f_1(X,Y)\circ Z}{f_1(Y,X)\circ Z}\cdot\frac{f_2(X,Y)\circ Z}{f_2(Y,X)\circ Z} = \frac{X\circ Z}{Y\circ Z}\cdot\frac{Y\circ Z}{X\circ Z} = 1\,. \tag{14}$$

Thus, the functions described in (12) and (13) satisfy the assumption (6).

Using the formula (8) for $m = 2$, f_1 and f_2 we get the following structure of price index:

$$I^p(Q^s,Q^t,P^s,P^t) = [\prod_{j=1}^{2} \frac{f_j(Q^s,Q^t)\circ P^t}{f_j(Q^s,Q^t)\circ P^s}]^{\frac{1}{2}} = \sqrt{\frac{Q^s\circ P^t}{Q^s\circ P^s}\cdot\frac{Q^t\circ P^t}{Q^t\circ P^s}} =$$

$$= \sqrt{I_L^p(Q^s,P^s,P^t)I_{Pa}^p(Q^t,P^s,P^t)} = I_F^p(Q^s,Q^t,P^s,P^t)\,. \tag{15}$$

The formula (15) leads to the following conclusions: the Fisher index is a special case of the generalized formula defined in (8), it means $I_F^p(Q^s,Q^t,P^s,P^t) \in \Lambda(2)$.Let us also notice that if we assume $m = 1$ and $f_j(X,Y) = \frac{1}{2}(X+Y)$,where $X = (x_1, x_2, .., x_N)$, $Y = (y_1, y_2, .., y_N)$, we have condition (6) satisfied and from (8) we get

$$I^p(Q^s,Q^t,P^s,P^t) = \frac{f_j(Q^s,Q^t)\circ P^t}{f_j(Q^s,Q^t)\circ P^s} = \frac{\frac{1}{2}(Q^s+Q^t)\circ P^t}{\frac{1}{2}(Q^s+Q^t)\circ P^s}\,. \tag{16}$$

The formula (16) is a definition of Lexis[1] index (see Zając (1994)), which is presented in the literature of the subject as

[1] The formula (17) is also called the Marhall-Edgeworth index.

$$I^p_{Lex}(Q^s,Q^t,P^s,P^t)=\frac{\sum_{i=1}^{N}\frac{q_i^s+q_i^t}{2}p_i^t}{\sum_{i=1}^{N}\frac{q_i^s+q_i^t}{2}p_i^s}. \tag{17}$$

Thus for (16) and (17) we have the additional conclusion: the Lexis index is a special case of the generalized formula defined in (8).

Taking $m=1$ and $f_1(X,Y)=<1,1,...,1>$ we get the unweighted index of Dutot (1738).

The formula (8) allows to create new definitions of aggregative price indexes. For example taking $m=4$, $f_1(X,Y)=\frac{1}{2}(X+Y)$, $f_2(X,Y)=f_1(X,Y)$, $f_3(X,Y)=X$ and $f_4(X,Y)=Y$ we get the following structure of price index (assumption (6) is certainly satisfied):

$$I^p_{New}(Q^s,Q^t,P^s,P^t)=[\prod_{j=1}^{4}\frac{f_j(Q^s,Q^t)\circ P^t}{f_j(Q^s,Q^t)\circ P^s}]^{\frac{1}{4}}=$$

$$=\sqrt[4]{\frac{0.5(Q^s+Q^t)\circ P^t}{0.5(Q^s+Q^t)\circ P^s}\cdot\frac{0.5(Q^s+Q^t)\circ P^t}{0.5(Q^s+Q^t)\circ P^s}\cdot\frac{Q^s\circ P^t}{Q^s\circ P^s}\cdot\frac{Q^t\circ P^t}{Q^t\circ P^s}}=$$

$$\sqrt[4]{(I^p_{Lex})^2(I^p_F)^2}=\sqrt{I^p_{Lex}I^p_F}. \tag{18}$$

Thus we have the conclusion, that the geometric mean of Fisher and Lexis indexes creates the new index, which also can be described by (8). Moreover, the following theorem is true:

Theorem 2

Let us assume that $I_1^P\in\Lambda(m_1)$ and $I_2^P\in\Lambda(m_2)$.

Let us also assume that $\exists k\in N \;\; m_2=km_1$.

Then we have $\sqrt{I_1^P I_2^P}\in\Lambda(2m_2)$.

Proof

The first assumption that $I_1^P\in\Lambda(m_1)$ means that

$$I_1^p(Q^s,Q^t,P^s,P^t)=[\prod_{j=1}^{m_1}\frac{f_{j1}(Q^s,Q^t)\circ P^t}{f_{j1}(Q^s,Q^t)\circ P^s}]^{\frac{1}{m_1}}, \tag{19}$$

and

$$\prod_{j=1}^{m_1} \frac{f_{j1}(Q^s,Q^t)\circ P^t}{f_{j1}(Q^t,Q^s)\circ P^t} = 1, \tag{20}$$

$$\prod_{j=1}^{m_1} \frac{f_{j1}(Q^s,Q^t)\circ P^s}{f_{j1}(Q^t,Q^s)\circ P^s} = 1. \tag{21}$$

Analogically, from the assumption $I_2^P \in \Lambda(m_2)$ we have

$$I_2^p(Q^s,Q^t,P^s,P^t) = [\prod_{j=1}^{m_2} \frac{f_{j2}(Q^s,Q^t)\circ P^t}{f_{j2}(Q^s,Q^t)\circ P^s}]^{\frac{1}{m_2}}, \tag{22}$$

and

$$\prod_{j=1}^{m_2} \frac{f_{j2}(Q^s,Q^t)\circ P^t}{f_{j2}(Q^t,Q^s)\circ P^t} = 1, \tag{23}$$

$$\prod_{j=1}^{m_2} \frac{f_{j2}(Q^s,Q^t)\circ P^s}{f_{j2}(Q^t,Q^s)\circ P^s} = 1. \tag{24}$$

Let us notice that in the case of $m_1 = m_2$ $(k=1)$, defining $\tilde{f}_j(X,Y) = f_{j1}(X,Y)$ and $\tilde{f}_{j+m_2}(X,Y) = f_{j2}(X,Y)$ for $j = 1,2,3,..,m_2$, we have

$$\sqrt{I_1^p I_2^p} = \sqrt{[\prod_{j=1}^{m_2} \frac{f_{j1}(Q^s,Q^t)\circ P^t}{f_{j1}(Q^s,Q^t)\circ P^s}]^{\frac{1}{m_2}} [\prod_{i=1}^{m_2} \frac{f_{i2}(Q^s,Q^t)\circ P^t}{f_{i2}(Q^s,Q^t)\circ P^s}]^{\frac{1}{m_2}}} =$$

$$= \sqrt{[\prod_{j=1}^{2m_2} \frac{\tilde{f}_j(Q^s,Q^t)\circ P^t}{\tilde{f}_j(Q^s,Q^t)\circ P^s}]^{\frac{1}{m_2}}} = [\prod_{j=1}^{2m_2} \frac{\tilde{f}_j(Q^s,Q^t)\circ P^t}{\tilde{f}_j(Q^s,Q^t)\circ P^s}]^{\frac{1}{2m_2}}, \tag{25}$$

where, from (20), (21), (23) and (24)

$$\prod_{j=1}^{2m_2} \frac{\tilde{f}_j(Q^s,Q^t)\circ P^t}{\tilde{f}_j(Q^t,Q^s)\circ P^t} = 1, \tag{26}$$

$$\prod_{j=1}^{2m_2} \frac{\tilde{f}_j(Q^s,Q^t)\circ P^s}{\tilde{f}_j(Q^t,Q^s)\circ P^s} = 1. \tag{27}$$

Thus, for $m_1 = m_2$ we have $\sqrt{I_1^P I_2^P} \in \Lambda(2m_2)$.

Let us consider the case when $m_2 = km_1$ for some $k \in N$. From (19) we get:

$$I_1^p(Q^s, Q^t, P^s, P^t) = [\prod_{j=1}^{m_1} \frac{f_{j1}(Q^s, Q^t) \circ P^t}{f_{j1}(Q^s, Q^t) \circ P^s}]^{\frac{1}{m_1}} =$$

$$= \{[\prod_{j=1}^{m_1} \frac{f_{j1}(Q^s, Q^t) \circ P^t}{f_{j1}(Q^s, Q^t) \circ P^s}]^k\}^{\frac{1}{km_1}} =$$

$$= [\prod_{j=1}^{km_1} \frac{\hat{f}_{j1}(Q^s, Q^t) \circ P^t}{\hat{f}_{j1}(Q^s, Q^t) \circ P^s}]^{\frac{1}{km_1}}, \tag{28}$$

where

$$\hat{f}_{j1} = f_{j1 - \left\lfloor \frac{j1-1}{m_1} \right\rfloor m_1} \quad \text{for } j1 = 1, 2, \ldots, km_1. \tag{29}$$

It is easy to check that then we have

$$\prod_{j=1}^{km_1} \frac{\hat{f}_{j1}(Q^s, Q^t) \circ P^t}{\hat{f}_{j1}(Q^t, Q^s) \circ P^t} = 1, \tag{30}$$

$$\prod_{j=1}^{km_1} \frac{\hat{f}_{j1}(Q^s, Q^t) \circ P^s}{\hat{f}_{j1}(Q^t, Q^s) \circ P^s} = 1. \tag{31}$$

From (28), (30) and (31) we have $I_1^p \in \Lambda(km_1)$.

Thus, taking into consideration that $I_2^p \in \Lambda(m_2)$ and $m_2 = km_1$, we have the thesis:

$$\sqrt{I_1^P I_2^P} \in \Lambda(2m_2).$$

Remark 1

By using the theorem 2 we can explain, that $\sqrt{I_{Lex}^p I_F^p} \in \Lambda(4)$ - see the formula (18).

In fact we have $I_{Lex}^p \in \Lambda(1)$ and $I_F^p \in \Lambda(2)$.

Taking $m_1 = 1$, $m_2 = 2$ and $k = 2$ we get from the theorem 2: $\sqrt{I_{Lex}^p I_F^p} \in \Lambda(2m_2)$.

The immediate consequence of the theorem 2 is the next theorem:

Theorem 3

Let us assume that $I_1^P \in \Lambda(m_1)$ and $I_2^P \in \Lambda(m_2)$. Then $\sqrt{I_1^P I_2^P} \in \Lambda(2m_1m_2)$.

The proof is immediate – it is enough to notice, that the following implication is true:

$I_2^p \in \Lambda(m_2) \Rightarrow I_2^p \in \Lambda(m_1m_2)$ (compare formulas (28) and (29)). Taking $k = m_2$ we can use the theorem 2, because $I_1^P \in \Lambda(m_1)$ and $I_2^P \in \Lambda(\tilde{m}_2)$, where $\tilde{m}_2 = km_1$.

Remark 2

The theorem 3 is described by the following implication:

$$I_1^P \in \Lambda(m_1) \ \wedge \ I_2^P \in \Lambda(m_2) \ \Rightarrow \ \sqrt{I_1^P I_2^P} \in \Lambda(2m_1m_2).$$

Let us notice that the opposite implication is not true: from (15) we know that $I_F^p(Q^s, Q^t, P^s, P^t) = \sqrt{I_L^p(Q^s, P^s, P^t) I_{Pa}^p(Q^t, P^s, P^t)} \in \Lambda(2)$, but Paasche and Laspeyres indexes do not satisfy the *time reversibility,* thus $I_L^p(Q^s, P^s, P^t) \notin \Lambda(1)$ and $I_{Pa}^p(Q^t, P^s, P^t) \notin \Lambda(1)$.

4. Conclusions

The presented, generalized formula for aggregative price indexes satisfies almost all postulates coming from axiomatic price index theory. Thus, we have a practical conclusion: it is easier to prove that a given index belongs to the considered class than verify all mentioned postulates. It is shown, that the ideal Fisher index and Lexis index are special cases of the proposed formula. Moreover, using the generalized formula we can easily define new price indexes, which also satisfy given postulates. The geometric mean of the indexes coming from the considered class generates the new index, also belonging to this class.

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